JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 1, 308–313 (1960)

# On the Stability of Solutions of the Linearized Plasma Equation

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We consider the equation

$$u_t + ixu - iaf'(x) \int_{-\infty}^{\infty} u\,dx = 0, \qquad u(x, 0) = h(x),$$

a linearized version of a basic equation in plasma physics. Deriving a relation equivalent to the usual dispersion relation, but of more convenient form, we derive some simple criteria for stability of solutions of this equation.

## I. INTRODUCTION

The equation

$$u_t + ixu - iaf'(x) \int_{-\infty}^{\infty} u\,dx, \qquad u(x, 0) = h(x), \tag{1.1}$$

is a linearized version of a basic equation in plasma physics. Here $u \equiv u(x, t)$, $f(x)$ is a real function of $x$ and $a$ is a positive constant. A problem of some importance is that of determining when the solutions of (1.1) are uniformly bounded as $t \to \infty$.

The usual approach to this problem in the physical literature is to write

$$u = e^{st}\phi(x), \tag{1.2}$$

where we normalize by the condition $\int_{-\infty}^{\infty}\phi(x)\,dx = 1$. Then (1.1) yields

$$\phi(x) = \frac{iaf'(x)}{s + ix}. \tag{1.3}$$

The normalization condition yields the "characteristic equation" for $s$,

$$ia \int_{-\infty}^{\infty} \frac{f'(x)\,dx}{s + ix} = 1, \tag{1.4}$$

called the "dispersion relation." Since it is generally rather difficult to study the distribution of the roots of an equation of this type, we wish to present a different approach which will enable us to present some simple conditions for stability.

## II. Reduction to Renewal Equation

Let us write (1.1) in the form

$$u_t + ixu = iaf'(x) \int_{-\infty}^{\infty} u\,dx, \qquad u(x, 0) = h(x), \tag{2.1}$$

and regard the right-hand side as a forcing term. Then we may write

$$u = e^{-ixt}h(x) + iaf'(x) \int_0^t e^{-ix(t-s)} \left[ \int_{-\infty}^{\infty} u(x_1, t)\,dx_1 \right] ds. \tag{2.2}$$

Integrating over $[-\infty, \infty]$, we have

$$\int_{-\infty}^{\infty} u\,dx = \int_{-\infty}^{\infty} e^{-ixt}h(x)\,dx + ia \int_0^t \left[ \int_{-\infty}^{\infty} f'(x)e^{-ix(t-s)}\,dx \right] \left[ \int_{-\infty}^{\infty} u\,dx_1 \right] ds. \tag{2.3}$$

Let us now introduce some new functions:

$$v(t) = \int_{-\infty}^{\infty} u(x, t)\,dx,$$

$$H(t) = \int_{-\infty}^{\infty} e^{-ixt}h(x)\,dx, \tag{2.4}$$

$$K(t) = \int_{-\infty}^{\infty} f'(x)e^{-ixt}\,dx.$$

Then

$$v(t) = H(t) + \int_0^t K(t-s)v(s)\,ds, \tag{2.5}$$

an integral equation of renewal type.

The Laplace transform readily yields the equation

$$L(v) = \frac{L(H)}{1 - L(K)}, \tag{2.6}$$

or, using the Laplace inverse,

$$v = \frac{1}{2\pi i}\int_C \frac{L(H)e^{st}\,ds}{1 - L(K)}, \tag{2.7}$$

where $C$ can be taken to be a line parallel to the imaginary axis and sufficiently far to the right.

The question of the asymptotic behavior of $u(x, t)$ as $t \to \infty$ thus reduces to the problem of determining the root with largest real part of the equation

$$1 = \int_0^\infty e^{-st}K(t)\,dt. \tag{2.8}$$

## III. Discussion

Using the definition of $K(t)$ given in (2.4), we see that (2.5) can be written

$$1 = \int_0^\infty e^{-st}\left[ia\int_{-\infty}^\infty f'(x)e^{-ixt}\,dx\right]dt$$

$$= ia\int_{-\infty}^\infty \frac{f'(x)\,dx}{s + ix}, \tag{3.1}$$

if we interchange the orders of integration. However, we definitely do not wish to do this. The advantage of using (2.5) rather than the dispersion relation lies in the fact that much more is known about the roots of equations of the form of (2.5) than the roots of equations such as (1.4). We shall illustrate this in the next section.

## IV. Some Simple Criteria

For the purpose of stability investigations, we wish to obtain simple conditions which ensure that (2.5) possesses no roots with nonnegative real part. Although a simple criterion is

$$\int_0^\infty |K(t)|\, dt < 1, \tag{4.1}$$

unless $K(t) \geqslant 0$ this is much too crude for useful application.

Let us suppose that $K(t)$ is real, and write $s = \sigma + i\tau$. Then (2.5) yields two equations,

$$1 = \int_0^\infty e^{-\sigma t} \cos t\tau K(t)\, dt,$$

$$0 = \int_0^\infty e^{-\sigma t} \sin t\tau K(t)\, dt. \tag{4.2}$$

Given a particular function $K(t)$ of reasonable analytic structure, these equations can be used to determine stability or nonstability.

A particular useful and simple criterion is the following

Lemma. *If $g(t) > 0$ and monotone decreasing, then*

$$\int_0^\infty g(t) \sin t\tau\, dt > 0 \tag{4.3}$$

*if* $\tau > 0$.

Write

$$\int_0^\infty g(t) \sin t\tau\, dt = \int_0^{\pi/\tau} \sin t\tau \left[ g(t) - g\left(t + \frac{\pi}{\tau}\right) + g\left(t + \frac{2\pi}{\tau}\right) - \dots \right] dt. \tag{4.4}$$

We see that the result actually holds under the weaker condition

$$g(t) - g\left(t + \frac{\pi}{\tau}\right) + g\left(t + \frac{2\pi}{\tau}\right) - g\left(t + \frac{3\pi}{\tau}\right) + \dots \geqslant 0 \tag{4.5}$$

for $0 \leqslant t \leqslant \pi/\tau$.

## V. An Application

As an application of the foregoing result, let us consider the important case where $f(x) = e^{-x^2}$. Since

$$\int_{-\infty}^{\infty} e^{-x^2 - ixt}\, dx = \sqrt{\pi} e^{-t^2/4} \tag{5.1}$$

we have

$$\int_{-\infty}^{\infty} (-2ixe^{-x^2}) e^{-ixt}\, dx = 2\sqrt{\pi} \frac{\partial}{\partial t} (e^{-t^2/4}) = -\sqrt{\pi} t e^{-t^2/4}. \tag{5.2}$$

Hence the equation (2.5) takes the form

$$1 = -a\sqrt{\pi} \int_0^{\infty} t e^{-st} e^{-t^2/4}\, dt. \tag{5.3}$$

Integration by parts yields

$$\int_0^{\infty} e^{-st} t e^{-t^2/4}\, dt = 2 - 2s \int_0^{\infty} e^{-st} e^{-t^2/4}\, dt. \tag{5.4}$$

Thus (5.3) becomes

$$\frac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi} s} = \int_0^{\infty} e^{-st} e^{-t^2/4}\, dt. \tag{5.5}$$

Write $s = \sigma + i\tau$ and equate real and complex parts. The resulting two equations are

$$\begin{aligned} \left[\frac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right] \frac{\sigma}{\sigma^2 + \tau^2} &= \int_0^{\infty} e^{-\sigma t} e^{-t^2/4} \cos t\tau\, dt, \\ -\left[\frac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right] \frac{\tau}{\sigma^2 + \tau^2} &= \int_0^{\infty} e^{-\sigma t} e^{-t^2/4} \sin t\tau\, dt. \end{aligned} \tag{5.6}$$

Without loss of generality take $\tau \geqslant 0$. Consider first the case $\tau > 0$.

Then if $\sigma \geqslant 0$, the function $e^{-\sigma t} e^{-t^2/4}$ is positive and monotone decreasing for $t \geqslant 0$. Hence the second equation in (5.6) cannot hold for a non-negative value of $\sigma$.

If $\tau = 0$, the first equation yields

$$\left[\frac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right] \frac{1}{\sigma} = \int_0^\infty e^{-\sigma t} e^{-t^2/4} \, dt, \tag{5.7}$$

a relation which holds for no value of $\sigma$ since

$$\frac{1}{\sigma} < \left[\frac{1 + 2a\sqrt{\pi}}{2a\sqrt{\pi}}\right] \frac{1}{\sigma} = \int_0^\infty e^{-\sigma t} e^{-t^2/4} \, dt < \int_0^\infty e^{-\sigma t} \, dt = \frac{1}{\sigma}. \tag{5.8}$$

In the case of general $f(x)$, we obtain criteria in terms of the Fourier transform of $f'(x)$, or of $f(x)$. In this way we can exhibit families of functions which yield stability.